

Observability Estimate and State Observation Problems for Stochastic Hyperbolic Equations*

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Abstract

In this paper, we derive a boundary and an internal observability inequality for stochastic hyperbolic equations with nonsmooth lower order terms. The required inequalities are obtained by global Carleman estimate for stochastic hyperbolic equations. By these inequalities, we study a state observation problem for stochastic hyperbolic equations. As a consequence, we also establish a unique continuation property for stochastic hyperbolic equations.

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1 Introduction

Let $T > 0$, $G \in \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with the C^2 boundary Γ . Let Γ_0 be a suitable chosen nonempty subset of Γ , whose definition will be given later. Put

$$Q \triangleq (0, T) \times G, \quad \Sigma \triangleq (0, T) \times \Gamma, \quad \Sigma_0 \triangleq (0, T) \times \Gamma_0, \\ \mathcal{O}_\delta(\Gamma_0) \triangleq \left\{ x \in G : \text{dist}(x, \Gamma_0) \leq \delta \right\} \text{ for some } \delta > 0.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined. Let H be a Banach space. Denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H -valued and $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$, by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all

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H -valued and $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted bounded processes, by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all H -valued and $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0, T]; H)}) < \infty$ (similarly, one can define $L^2_{\mathcal{F}}(\Omega; C^k([0, T]; H))$ for any positive integer k), all of these spaces are endowed with the canonical norm.

Throughout this paper, we make the following assumptions on the coefficients $b^{ij} \in C^1(G)$:

1. $b^{ij} = b^{ji}$ ($i, j = 1, 2, \dots, n$);
2. For some constant $s_0 > 0$,

$$\sum_{i,j} b^{ij} \xi^i \xi^j \geq s_0 |\xi|^2, \quad \forall (x, \xi) \triangleq (x, \xi^1, \dots, \xi^n) \in G \times \mathbb{R}^n. \quad (1.1)$$

Here and in what follows, we denote $\sum_{i,j=1}^n$ simply by $\sum_{i,j}$. For simplicity, we use the notation

$y_i \equiv y_i(x) \triangleq \frac{\partial y(x)}{\partial x_i}$, where x_i is the i -th coordinate of a generic point $x = (x_1, \dots, x_n)$ in \mathbb{R}^n . In a similar manner, we use notations z_i, v_i , etc. for the partial derivatives of z and v with respect to x_i . Also, we denote by $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$ the unit outward normal vector of Γ at point x .

Let us consider the following stochastic hyperbolic equation:

$$\begin{cases} dz_t - \sum_{i,j} (b^{ij} z_i)_j dt = [b_1 z_t + b_2 \cdot \nabla z + b_3 z + f] dt + (b_4 z + g) dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0, z_t(0) = z_1 & \text{in } G. \end{cases} \quad (1.2)$$

Here the initial data $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$, the coefficients b_i ($1 \leq i \leq 4$) satisfy that

$$\begin{aligned} b_1 &\in L^\infty_{\mathcal{F}}(0, T; L^\infty(G)), & b_2 &\in L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n)), \\ b_3 &\in L^\infty_{\mathcal{F}}(0, T; L^p(G)) \ (p \in [n, \infty]), & b_4 &\in L^\infty_{\mathcal{F}}(0, T; L^\infty(G)), \end{aligned} \quad (1.3)$$

and nonhomogeneous terms

$$f \in L^2_{\mathcal{F}}(0, T; L^2(G)), \quad g \in L^2_{\mathcal{F}}(0, T; L^2(G)). \quad (1.4)$$

Put

$$H_T \triangleq L^2_{\mathcal{F}}(\Omega; C([0, T]; H_0^1(G))) \cap L^2_{\mathcal{F}}(\Omega; C^1([0, T]; L^2(G))). \quad (1.5)$$

Clearly, H_T is a Banach space with the canonical norm.

Now we give the definition of the solution to the equation (1.2).

Definition 1.1 *We call $z \in H_T$ a solution to the equation (1.2) if the following two conditions hold:*

1. $z(0) = z_0$ in G , P -a.s., and $z_t(0) = z_1$ in G , P -a.s.
2. For any $t \in [0, T]$ and any $\eta \in H_0^1(G)$, it holds that

$$\begin{aligned}
& \int_G z_t(t, x) \eta(x) dx - \int_G z_t(0, x) \eta(x) dx \\
&= \int_0^t \int_G \left\{ - \sum_{i,j} b^{ij}(x) z_i(s, x) \eta_j(x) + [b_1(s, x) z_t(s, x) + b_2(s, x) \cdot \nabla z(s, x) \right. \\
&\quad \left. + b_3(s, x) z(s, x) + f(s, x)] \eta(x) \right\} dx ds \\
&\quad + \int_0^t \int_G [b_4(s, x) z(s, x) + g(s, x)] \eta(x) dx dB(s), \quad P\text{-a.s.}
\end{aligned} \tag{1.6}$$

For any initial data $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$, one can show that the equation (1.2) admits a unique solution $z \in H_T$ (see [25] for details).

Before giving Γ_0 , we introduce the following condition:

Condition 1.1 *There exists a positive function $d(\cdot) \in C^2(\overline{G})$ satisfying the following:*

1. For some constant $\mu_0 > 0$, it holds

$$\begin{aligned}
& \sum_{i,j} \left\{ \sum_{i',j'} \left[2b^{ij'}(b^{i'j} d_{i'})_{j'} - b_{j'}^{ij} b^{i'j'} d_{i'} \right] \right\} \xi^i \xi^j \geq \mu_0 \sum_{i,j} b^{ij} \xi^i \xi^j, \\
& \forall (x, \xi^1, \dots, \xi^n) \in \overline{G} \times \mathbb{R}^n.
\end{aligned} \tag{1.7}$$

2. There is no critical point of $d(\cdot)$ in \overline{G} , i.e.,

$$\min_{x \in \overline{G}} |\nabla d(x)| > 0. \tag{1.8}$$

Remark 1.1 *If $(b^{ij})_{1 \leq i,j \leq n}$ is the identity matrix, then $d(x) = |x - x_0|^2$ satisfies Condition 1.1, where x_0 is any point which belongs to $\mathbb{R}^n \setminus \overline{G}$.*

Remark 1.2 *Condition 1.1 was first given in [5] for the purpose of obtaining an internal observability estimate for hyperbolic equations. In that paper, the authors also gave some explanation of Condition 1.1 and some interesting nontrivial examples satisfying it. Further, a detailed study of this condition is given in [13].*

The Γ_0 is as follows:

$$\Gamma_0 \triangleq \left\{ x \in \Gamma \mid \sum_{i,j} b^{ij} d_i(x) \nu^j(x) > 0 \right\}. \tag{1.9}$$

It is easy to check that if $d(\cdot)$ satisfies Condition 1.1, then for any given constants $a \geq 1$ and $b \in \mathbb{R}$, the function $\tilde{d} = ad + b$ still satisfies Condition 1.1 with μ_0 replaced by $a\mu_0$. Therefore we may choose d , μ_0 , $c_0 > 0$, $c_1 > 0$ and T to such that the following condition holds:

Condition 1.2

1. $\frac{1}{4} \sum_{i,j} b^{ij}(x) d_i(x) d_j(x) \geq R_1^2 \triangleq \max_{x \in \overline{G}} d(x) \geq R_0^2 \triangleq \min_{x \in \overline{G}} d(x), \quad \forall x \in \overline{G}. \quad (1.10)$
2. $T > T_0 \triangleq 2R_1.$
3. $\left(\frac{2R_1}{T}\right)^2 < c_1 < \frac{2R_1}{T}.$
4. $\mu_0 - 4c_1 - c_0 > 0.$

Remark 1.3 As we have explained, since $\sum_{i,j} b^{ij} d_i d_j > 0$, and one can choose μ_0 in Condition 1.1 large enough, Condition 1.2 could be satisfied obviously. We put it here just in order to emphasize the relationship between $0 < c_0 < c_1 < 1$, μ_0 and T .

Remark 1.4 If $(b^{ij})_{1 \leq i,j \leq n}$ is the identity matrix, then it is easy to show that

$$d(x) = 2|x - x_0|^2$$

for some $x_0 \notin \overline{G}$ satisfy (1.7) and (1.8) in Condition 1.1. However, this $d(\cdot)$ does not satisfy (1.10) in the Condition 1.2. On the other hand, if we consider the problem with $(b^{ij})_{1 \leq i,j \leq n} = \text{diag}(1, 1, \dots, 1)$, we do not need (1.10). Indeed, in this case, the inequality (1.12) and (1.13) below hold for all $T > 2 \max_{x \in \overline{G}} |x - x_0|$. One can follow the proofs of Theorem 1.1 and 1.2 to see this. We omit the details.

In the rest of this paper, we use C to denote a generic positive constant depending on G , T , Γ_0 , $b^{ij}(i, j = 1, \dots, n)$, d , c_0 and c_1 (unless otherwise stated), which may change from line to line.

Put

$$r_1 \triangleq |b_2|_{L_{\mathcal{F}}^\infty(0,T;L^\infty(G;\mathbb{R}^n))} + |(b_1, b_4)|_{L_{\mathcal{F}}^\infty(0,T;(L^\infty(G))^2)} \quad \text{and} \quad r_2 = |b_3|_{L_{\mathcal{F}}^\infty(0,T;L^p(G))}. \quad (1.11)$$

Now we give our main results. The first one is the boundary observability estimate for the equation (1.2).

Theorem 1.1 *Let Condition 1.1 and Condition 1.2 be satisfied. For any solution of the equation (1.2), we have*

$$\begin{aligned} & |(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} \\ & \leq C e^{C(r_1^2 + r_2^{\frac{1}{3/2 - n/p} + 1})} \left(\left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0,T;L^2(\Gamma_0))} + |f|_{L_{\mathcal{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathcal{F}}^2(0,T;L^2(G))} \right). \end{aligned} \quad (1.12)$$

The second one is the internal observability estimate for the equation (1.2).

Theorem 1.2 *Let Condition 1.1 and Condition 1.2 be satisfied. For any solution of the equation (1.2), it holds*

$$\begin{aligned} & |(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} \\ & \leq e^{C(r_1^2 + r_2^{\frac{1}{3/2-n/p}+1})} \left(|\nabla z|_{L_{\mathcal{F}}^2(0,T;L^2(\mathcal{O}_\delta(\Gamma_0)))} + |f|_{L_{\mathcal{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathcal{F}}^2(0,T;L^2(G))} \right). \end{aligned} \quad (1.13)$$

Remark 1.5 *Inequality (1.12) resp. ((1.13)) is referred to as observability estimate since it provides a quantitative estimate of the norm of the initial data in terms of the observed quantity, by means of the observability constant C . Indeed, the inequality (1.12) resp. ((1.13)) allows one to estimate the total energy of solutions at time 0 in terms of the partial energy localized in the observation subboundary Γ_0 (resp. the observation subdomain \mathcal{O}_δ). This sort of inequality is strongly relevant to control problems and state observation problems for stochastic hyperbolic equations.*

Remark 1.6 *Compared with inequality (1.13), it is more interesting to establish the following inequality:*

$$\begin{aligned} & |(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; L^2(G) \times H^{-1}(G))} \\ & \leq e^{C(r_1^2 + r_2^{\frac{1}{3/2-n/p}+1})} \left(|z|_{L_{\mathcal{F}}^2(0,T;L^2(\mathcal{O}_\delta(\Gamma_0)))} + |f|_{L_{\mathcal{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathcal{F}}^2(0,T;L^2(G))} \right). \end{aligned} \quad (1.14)$$

However, we do not know how to obtain this result now.

Thanks to its important applications in Control Theory and Inverse Problems for hyperbolic equations, and to its strong connection with the unique continuation for solutions to hyperbolic equations, the observability estimate for hyperbolic equations have been studied extensively in the literature. There are four main approaches to study it. The first one is the multiplier techniques (see [12] for example). The second one is nonharmonic Fourier series technique (see [10] for example). The third one is based on the Microlocal Analysis (see [2] for example). The last one is the global Carleman estimate (see [4, 5] for example).

Among the above four methods, the global Carleman estimate is the most common and powerful technique to derive observability inequalities. It can be regarded as a more developed version of the classical multiplier technique. Compared with the multiplier method, the Carleman approach is robust with respect to the lower order terms. Compared with the microlocal analysis, it requires less regularity on coefficients and domain. Compared with the nonharmonic Fourier series method, it has much less restrictions on the shape of the domain.

There are very few works addressing the observability problems for stochastic partial differential equations. To the best of our knowledge, [1, 16, 18, 25] are the only references for this topic. In [1, 18] the observability estimate for stochastic heat equations is studied. [16] is devoted to the observability estimate for stochastic Schrödinger equations while [25] is concerned with the observability estimate for stochastic wave equations. In [25], the author proves a boundary observability estimate for the equation (1.2) with $(b^{ij})_{1 \leq i,j \leq n}$ being the identity matrix. More precisely, the author proves that

$$\begin{aligned} & |(z(t), z_t(t))|_{L^2(\Omega, \mathcal{F}_t, P; H_0^1(G) \times L^2(G))} \\ & \leq C e^{Ct^{-1}\mathcal{A}} \left(\left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0,T;L^2(\Gamma_0))} + |f|_{L_{\mathcal{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathcal{F}}^2(0,T;L^2(G))} \right), \end{aligned} \quad (1.15)$$

where z solves the equation (1.2), T satisfies

$$\frac{(4 + 5c) \min_{x \in \overline{G}} |x - x_0|^2}{9c} > c^2 T^2 > 4 \max_{x \in \overline{G}} |x - x_0|^2 \quad (1.16)$$

for some $c \in (0, 1)$ and $x_0 \in \mathbb{R}^n \setminus \overline{G}$, and

$$\mathcal{A} = \mathcal{A}(b_1, b_2, b_3, b_4) \triangleq |b_2|_{L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n))}^2 + |(b_1, b_4)|_{L^\infty_{\mathcal{F}}(0, T; L^\infty(G))^2}^2 + |b_3|_{L^\infty_{\mathcal{F}}(0, T; L^n(G))}^2.$$

There are three main differences between the inequality (1.12) and (1.15). The first is that the left-hand side of (1.12) is $|(z_0, z_1)|_{H_0^1(G) \times L^2(G)}$. However, it seems that one cannot simply replace $|(z(t), z_t(t))|_{L^2(\Omega, \mathcal{F}_t, P; H_0^1(G) \times L^2(G))}$ by $|(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))}$ directly in (1.15), thanks to the term $e^{Ct^{-1}\mathcal{A}}$. Although by Proposition 2.2 in this paper, one can get the estimate for $|(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))}$ by (1.15), there are other two differences. The second is that the observation time T in (1.15) should satisfy (1.16), which is usually much more restrictive than that $T > 2 \max_{x \in \overline{G}} |x - x_0|$ for our result (see Remark 1.4). The third is that the observability constant in (1.15) is not as sharp as that in (1.12). Indeed, it is clear that

$$|b_1|_{L^\infty(Q)}^2 + |b_2|_{L^\infty(Q; \mathbb{R}^n)}^2 + |b_3|_{L^\infty(0, T; L^p(G))}^{\frac{1}{3/2 - n/p}} \leq \mathcal{A}(b_1, b_2, b_3, 0) + C \quad \forall p \in [n, \infty].$$

The rest of this paper is organized as follows. In Section 2, we collect some preliminaries. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2. Section 5 is addressed to a state observation problem of semilinear stochastic hyperbolic equations. At last, in Section 6, we present some further comments and open problems.

2 Some Preliminaries

In this section, we present some preliminary results. First, we give a hidden regularity property of solutions to the equation (1.2).

Proposition 2.1 *For any solution of the equation (1.2), it holds that*

$$\begin{aligned} & \left| \frac{\partial z}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))} \\ & \leq e^{C(r_1^2 + r_2^2 + 1)} \left(|(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} + |f|_{L^2_{\mathcal{F}}(0, T; L^2(G))} + |g|_{L^2_{\mathcal{F}}(0, T; L^2(G))} \right). \end{aligned} \quad (2.1)$$

Remark 2.1 *In [25], the author proved Proposition 2.1 for the case $(b^{ij})_{1 \leq i, j \leq n}$ is identity matrix. The proof Lemma 2.1 for general $(b^{ij})_{1 \leq i, j \leq n}$ is similar. We only give a sketch of it here.*

Proof: For any

$$h \triangleq (h^1, \dots, h^n) \in C^1(\mathbb{R}_t \times \mathbb{R}_x^n; \mathbb{R}^n),$$

by direct computation, we can show that

$$\begin{aligned}
& - \sum_{i=1}^n \left[2(h \cdot \nabla z) \sum_{j=1}^n b^{ij} z_{x_j} + h^i \left(z_t^2 - \sum_{i,j=1}^n b^{ij} z_{x_i} z_{x_j} \right) \right]_{x_i} dt \\
& = 2 \left[\left(dz_t - \sum_{i,j=1}^n (b^{ij} z_{x_i})_{x_j} dt \right) h \cdot \nabla z - d(z_t h \cdot \nabla z) + z_t h_t \cdot \nabla z dt - \sum_{i,j,k=1}^n b^{ij} z_{x_i} z_{x_k} h_{x_j}^k dt \right] \\
& \quad - (\operatorname{div} h) z_t^2 dt + \sum_{i,j=1}^n z_{x_j} z_{x_i} \operatorname{div} (b^{ij} h) dt.
\end{aligned} \tag{2.2}$$

Since $\Gamma \in C^2$, one can find a vector field $\xi = (\xi^1, \dots, \xi^n) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\xi = \nu$ on Γ (see [9, page 18]). Setting $h = \xi$ in the equality (2.2), integrating (2.2) in Q , taking expectation in Ω and integrating by parts, we get inequality (2.1) immediately. \square

Further, we give an energy estimate for the equation (1.2), which plays an important role in the proof of the observability estimate.

Proposition 2.2 *For any z solves the equation (1.2), it holds that*

$$\begin{aligned}
& \mathbb{E} \int_G (|z_t(t, x)|^2 + |\nabla z(t, x)|^2) dx \\
& \leq C e^{C \left(r_1^2 + r_2^{\frac{1}{2-n/p} + 1} \right) T} \mathbb{E} \int_G (|z_t(s, x)|^2 + |\nabla z(s, x)|^2) dx \\
& \quad + C \mathbb{E} \int_0^T \int_G [f^2(\tau, x) + g^2(\tau, x)] dx d\tau \Big\},
\end{aligned} \tag{2.3}$$

for any $0 \leq s, t \leq T$.

Proof: Without loss of generality, we assume that $t \leq s$. Let

$$\mathcal{E}(t) = \mathbb{E} \int_G \left[|z_t(t, x)|^2 + |\nabla z(t, x)|^2 + r_2^{\frac{2}{2-n/p}} |z(t, x)|^2 \right] dx.$$

From Poincaré's inequality, we get

$$\mathbb{E} \int_G (|z_t(t)|^2 + |\nabla z(t)|^2) dx \leq \mathcal{E}(t) \leq C \left(r_2^{\frac{2}{2-n/p}} + 1 \right) \mathbb{E} \int_G (|z_t(t)|^2 + |\nabla z(t)|^2) dx. \tag{2.4}$$

By means of Itô's formula, we have

$$d(z_t^2) = 2z_t dz_t + (dz_t)^2,$$

which implies that

$$\begin{aligned}
& \mathbb{E} \int_G \left(|z_t(s, x)|^2 + r_2^{\frac{2}{2-n/p}} |z(s, x)|^2 \right) dx - \mathbb{E} \int_G \left(|z_t(t, x)|^2 + r_2^{\frac{2}{2-n/p}} |z(t, x)|^2 \right) dx \\
&= -2\mathbb{E} \int_t^s \int_G \sum_{i,j} b^{ij}(x)(\tau, x) z_i(\tau, x) z_{jt}(\tau, x) dx d\tau + \mathbb{E} \int_t^s \int_G z_t(\tau, x) \left[b_1(\tau, x) z_t(\tau, x) \right. \\
&\quad \left. + 2b_2(\tau, x) \cdot \nabla z(\tau, x) + b_3(\tau, x) z(\tau, x) + f(\tau, x) \right] dx d\tau \\
&\quad + \mathbb{E} \int_t^s \int_G [b_4(\tau, x) z(\tau, x) + g(\tau, x)]^2 dx d\tau + 2r_2^{\frac{2}{2-n/p}} \mathbb{E} \int_t^s \int_G z_t(\tau, x) z(\tau, x) dx d\tau.
\end{aligned} \tag{2.5}$$

Therefore, we obtain that

$$\begin{aligned}
& \mathbb{E} \int_G \left[|z_t(s, x)|^2 + \sum_{i,j} b^{ij}(x) z_i(s, x) z_j(s, x) \right] dx \\
& - \mathbb{E} \int_G \left[|z_t(t, x)|^2 + \sum_{i,j} b^{ij}(x) z_i(t, x) z_j(t, x) \right] dx \\
&= \mathbb{E} \int_t^s \int_G z_t(\tau, x) \left[b_1(\tau, x) z_t(\tau, x) + b_2(\tau, x) \cdot \nabla z(\tau, x) + b_3(\tau, x) z(\tau, x) + f(\tau, x) \right] dx d\tau \\
&\quad + \mathbb{E} \int_t^s \int_G [b_4(\tau, x) z(\tau, x) + g(\tau, x)]^2 dx d\tau + 2r_2^{\frac{2}{2-n/p}} \mathbb{E} \int_t^s \int_G z_t(\tau, x) z(\tau, x) dx d\tau \\
&\leq C(r_1^2 + 1) \mathbb{E} \int_t^s \int_G [z_t^2(\tau, x) + |\nabla z(\tau, x)|^2 + z^2(\tau, x)] dx d\tau + 2r_2^{\frac{2}{2-n/p}} \mathbb{E} \int_t^s \int_G z_t(\tau, x) z(\tau, x) dx d\tau \\
&\quad + \mathbb{E} \int_t^s \int_G b_3(\tau, x) z(\tau, x) z_t(\tau, x) dx d\tau + 2\mathbb{E} \int_t^s \int_G [f^2(\tau, x) + g^2(\tau, x)] dx d\tau.
\end{aligned} \tag{2.6}$$

Put $p_1 = \frac{2p}{n-2}$ and $p_2 = \frac{2p}{p-n}$. It is easy to check that

$$\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{2} = 1 \quad \text{and} \quad \frac{1}{2(n/p)^{-1}} + \frac{1}{2(1-n/p)^{-1}} + \frac{1}{2} = 1.$$

By Hölder's inequality and Sobolov's embedding theorem, we find

$$\begin{aligned}
& \left| \mathbb{E} \int_G b_3(\tau, x) z(\tau, x) z_t(\tau, x) dx \right| \\
&\leq \mathbb{E} \int_G |b_3(\tau, x)| |z(\tau, x)|^{\frac{n}{p}} |z(\tau, x)|^{1-\frac{n}{p}} |z_t(\tau, x)| dx \\
&\leq r_2 \mathbb{E} \left(\left\| |z(\tau, \cdot)|^{\frac{n}{p}} \right\|_{L^{p_1}(G)} \left\| |z(\tau, \cdot)|^{1-\frac{n}{p}} \right\|_{L^{p_2}(G)} \left\| z_t(\tau, \cdot) \right\|_{L^2(G)} \right) \\
&= r_2 \mathbb{E} \left(\left\| |z(\tau, \cdot)|^{\frac{n}{p}} \right\|_{L^{\frac{p}{n-2}}(G)} \left\| |z(\tau, \cdot)|^{1-\frac{n}{p}} \right\|_{L^2(G)} \left\| z_t(\tau, \cdot) \right\|_{L^2(G)} \right) \\
&= r_2^{\frac{1}{2-n/p}} \mathbb{E} \left(\left\| |z(\tau, \cdot)|^{\frac{n}{p}} \right\|_{L^{\frac{p}{n-2}}(G)} r_2^{\frac{1-n/p}{2-n/p}} \left\| |z(\tau, \cdot)|^{1-\frac{n}{p}} \right\|_{L^2(G)} \left\| z_t(\tau, \cdot) \right\|_{L^2(G)} \right).
\end{aligned} \tag{2.7}$$

Since

$$\left\{ \begin{array}{l} |z(\tau, \cdot)|_{L^{\frac{n}{n-2}}(G)}^{\frac{n}{p}} \leq \left[\int_G \left(|z_t(\tau, x)|^2 + |\nabla z(\tau, x)|^2 + r_2^{\frac{2}{2-n/p}} |z(\tau, x)|^2 \right) dx \right]^{\frac{n}{2p}}, \\ r_2^{\frac{1-n/p}{2-n/p}} |z(\tau, \cdot)|_{L^2(G)}^{1-\frac{n}{p}} \leq \left[\int_G \left(|z_t(\tau, x)|^2 + |\nabla z(\tau, x)|^2 + r_2^{\frac{2}{2-n/p}} |z(\tau, x)|^2 \right) dx \right]^{\frac{1}{2}-\frac{n}{2p}}, \\ |z_t(\tau, \cdot)|_{L^2(G)} \leq \left[\int_G \left(|z_t(\tau, x)|^2 + |\nabla z(\tau, x)|^2 + r_2^{\frac{2}{2-n/p}} |z(\tau, x)|^2 \right) dx \right]^{\frac{1}{2}}, \end{array} \right.$$

from (2.7), we get

$$\left| \mathbb{E} \int_G b_3(\tau, x) z(\tau, x) z_t(\tau, x) dx \right| \leq r_2^{\frac{1}{2-n/p}} \mathcal{E}(\tau). \quad (2.8)$$

By a similar argument, we obtain that

$$\begin{aligned} & r_2^{\frac{2}{2-n/p}} \mathbb{E} \int_G z(\tau, x) z_t(\tau, x) dx \\ & \leq \frac{1}{2} r_2^{\frac{1}{2-n/p}} \mathbb{E} \int_G \left(r_2^{\frac{2}{2-n/p}} z^2(\tau, x) + z_t^2(\tau, x) \right) dx \\ & \leq \frac{1}{2} r_2^{\frac{1}{2-n/p}} \mathcal{E}(\tau). \end{aligned} \quad (2.9)$$

From (2.8), (2.9) and the property of $b^{ij}(i, j = 1, \dots, n)$ (see (1.1)), we find that

$$\mathcal{E}(t) \leq C \left\{ \mathcal{E}(s) + \left(r_1^2 + r_2^{\frac{1}{2-n/p}} + 1 \right) \int_t^s \mathcal{E}(\tau) d\tau + \mathbb{E} \int_t^s \int_G [f^2(\tau, x) + g^2(\tau, x)] dx d\tau \right\}. \quad (2.10)$$

This, together with backward Gronwall's inequality, implies that

$$\mathcal{E}(t) \leq e^{C \left(r_1^2 + r_2^{\frac{1}{2-n/p}} + 1 \right) (s-t)} \mathcal{E}(s) + C \mathbb{E} \int_t^s \int_G [f^2(\tau, x) + g^2(\tau, x)] dx d\tau. \quad (2.11)$$

From (2.4) and (2.11), we get

$$\begin{aligned} & \mathbb{E} \int_G \left(|z_t(t, x)|^2 + |\nabla z(t, x)|^2 \right) dx \\ & \leq C e^{C \left(r_1^2 + r_2^{\frac{1}{2-n/p}} + 1 \right) (s-t)} \mathbb{E} \int_G \left(|z_t(s, x)|^2 + |\nabla z(s, x)|^2 \right) dx \\ & \quad + C \mathbb{E} \int_t^s \int_G [f^2(\tau, x) + g^2(\tau, x)] dx d\tau, \end{aligned} \quad (2.12)$$

which leads to the inequality (2.3) immediately. \square

At last, we introduce the following known result, which plays a key role in getting the boundary and internal Carleman estimate.

Lemma 2.1 [25, Theorem 4.1] Let $p^{ij} \in C^1((0, T) \times \mathbb{R}^n)$ satisfy

$$p^{ij} = p^{ji}, \quad i, j = 1, 2, \dots, n,$$

$l, \Psi \in C^2((0, T) \times \mathbb{R}^n)$. Assume that u is an $H_{loc}^2(\mathbb{R}^n)$ -valued and $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process such that u_t is an $L^2(\mathbb{R}^n)$ -valued semimartingale. Set $\theta = e^l$ and $v = \theta u$. Then, for a.e. $x \in \mathbb{R}^n$ and P -a.s. $\omega \in \Omega$,

$$\begin{aligned} & \theta \left(-2l_t v_t + 2 \sum_{i,j} p^{ij} l_i v_j + \Psi v \right) \left[du_t - \sum_{i,j} (p^{ij} u_i)_j dt \right] \\ & + \sum_{i,j} \left[\sum_{i',j'} (2p^{ij} p^{i'j'} l_{i'} v_i v_{j'} - p^{ij} p^{i'j'} l_i v_{i'} v_{j'}) - 2p^{ij} l_t v_i v_t + p^{ij} l_i v_t^2 \right. \\ & \quad \left. + \Psi p^{ij} v_i v - \left(Al_i + \frac{\Psi_i}{2} \right) p^{ij} v^2 \right]_j dt \\ & + d \left[\sum_{i,j} p^{ij} l_t v_i v_j - 2 \sum_{i,j} p^{ij} l_i v_j v_t + l_t v_t^2 - \Psi v_t v + \left(Al_t + \frac{\Psi_t}{2} \right) v^2 \right] \\ & = \left\{ \left[l_{tt} + \sum_{i,j} (p^{ij} l_i)_j - \Psi \right] v_t^2 - 2 \sum_{i,j} [(p^{ij} l_j)_t + p^{ij} l_{tj}] v_i v_t \right. \\ & \quad + \sum_{i,j} \left[(p^{ij} l_t)_t + \sum_{i',j'} (2p^{ij'} (p^{i'j} l_{i'})_{j'} - (p^{ij} p^{i'j'} l_{i'})_{j'}) + \Psi p^{ij} \right] v_i v_j \\ & \quad \left. + B v^2 + \left(-2l_t v_t + 2 \sum_{i,j} p^{ij} l_i v_j + \Psi v \right)^2 \right\} dt + \theta^2 l_t (du_t)^2, \end{aligned} \quad (2.13)$$

where $(du_t)^2$ denotes the quadratic variation process of u_t , A and B are stated as follows:

$$\begin{cases} A \triangleq (l_t^2 - l_{tt}) - \sum_{i,j} (p^{ij} l_i l_j - p_j^{ij} l_i - p^{ij} l_{ij}) - \Psi, \\ B \triangleq A\Psi + (Al_t)_t - \sum_{i,j} (Ap^{ij} l_i)_j + \frac{1}{2} \left[\Psi_{tt} - \sum_{i,j} (p^{ij} \Psi_i)_j \right]. \end{cases} \quad (2.14)$$

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by means of the global Carleman estimate. In what follows, for $\lambda \in \mathbb{R}$, we use $O(\lambda^r)$ to denote a function of order λ^r for large λ .

Proof of Theorem 1.1: We divide the proof into three steps.

Step 1. On one hand, from Condition 1.2, we know that there is an $\varepsilon_1 \in (0, 1/2)$ such that

$$l(t, x) \leq \lambda \left(\frac{R_1^2}{2} - \frac{cT^2}{8} \right) < 0, \quad \forall (t, x) \in \left[\left(0, \frac{T}{2} - \varepsilon_1 T \right) \cup \left(\frac{T}{2} - \varepsilon_1 T, T \right) \right] \times G. \quad (3.1)$$

On the other hand, since

$$l\left(\frac{T}{2}, x\right) = d(x) \geq R_0^2, \quad \forall x \in G,$$

we can find an $\varepsilon_0 \in (0, \varepsilon_1)$ such that

$$l(t, x) \geq \frac{R_0^2}{2}, \quad \forall (t, x) \in \left(\frac{T}{2} - \varepsilon_0 T, \frac{T}{2} + \varepsilon_0 T\right) \times G. \quad (3.2)$$

Now we choose a $\chi \in C_0^\infty[0, T]$ satisfying

$$\chi = 1 \text{ in } \left(\frac{T}{2} - \varepsilon_1 T, \frac{T}{2} + \varepsilon_1 T\right). \quad (3.3)$$

Let $y = \chi z$ for z solving the equation (1.2), then we know that y is a solution to the following equation:

$$\begin{cases} dy_t - \sum_{i,j} (b^{ij} y_i)_j dt = [b_1 y_t + (b_2, \nabla y) + b_3 y + \chi f + \alpha] dt + (b_4 y + \chi g) dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y(T) = 0, y_t(0) = y_t(T) = 0 & \text{in } G. \end{cases} \quad (3.4)$$

Here $\alpha = \chi_{tt} z + 2\chi_t z_t - b_1 \chi_t z$.

Step 2. We apply Lemma 2.1 to the solution of the equation (3.4). In the present case, we choose

$$p^{ij} = b^{ij}, \Psi = l_{tt} + \sum_{i,j} (b^{ij} l_i)_j - \lambda c_0,$$

and then estimate the terms in (2.13) one by one.

We first analyze the terms which stand for the “energy” of the solution. The point is to compute the order of λ in the coefficients of $|v_t|^2$, $|\nabla v|^2$ and $|v|^2$. Clearly, the term for $|v_t|^2$ reads

$$\left\{ l_{tt} + \sum_{i,j} (b^{ij} l_i)_j - \Psi \right\} v_t^2 = \lambda c_0 v_t^2. \quad (3.5)$$

Noting that $b^{ij} (1 \leq i, j \leq n)$ are independent of t and $l_{tj} = l_{jt} = 0$, we get that

$$\sum_{i,j} [(b^{ij} l_j)_t + b^{ij} l_{tj}] v_i v_t = 0. \quad (3.6)$$

By Condition 1.1 and Condition 1.2, we have that

$$\begin{aligned} & \sum_{i,j} \left\{ (b^{ij} l_t)_t + \sum_{i',j'} [2b^{ij'} (b^{i'j} l_{i'})_{j'} - (b^{ij} b^{i'j'} l_{i'})_{j'}] + \Psi b^{ij} \right\} v_i v_j \\ &= \sum_{i,j} \left\{ 2b^{ij} l_{tt} - b^{ij} \lambda c_0 + \sum_{i',j'} [2b^{ij'} (b^{i'j} l_{i'})_{j'} - b_{j'}^{ij} b^{i'j'} l_{i'}] \right\} v_i v_j \\ &\geq \lambda (\mu_0 - 4c_1 - c_0) \sum_{i,j} b^{ij} v_i v_j. \end{aligned}$$

Now we compute the coefficients of $|v|^2$.

$$\begin{aligned}
A &= l_t^2 - l_{tt} - \sum_{i,j} [b^{ij} l_i l_j - (b^{ij} l_i)_j] - \Psi \\
&= \lambda^2 c_1^2 (2t - T)^2 + 4\lambda c_1 + \lambda c_0 - \sum_{i,j} b^{ij} l_i l_j \\
&= \lambda^2 \left[c_1^2 (2t - T)^2 - \sum_{i,j} b^{ij} d_i d_j \right] + O(\lambda).
\end{aligned}$$

By the definition of B , we see that

$$\begin{aligned}
B &= A\Psi + (Al_t)_t - \sum_{i,j} (Ab^{ij} l_i)_j + \frac{1}{2} \sum_{i,j} [\Psi_{tt} - (b^{ij} \Psi_i)_j] \\
&= 2Al_{tt} - \lambda c_0 A - \sum_{i,j} b^{ij} l_i A_j + A_t l_t - \frac{1}{2} \sum_{i,j} \sum_{i',j'} [b^{ij} (b^{i'j'} l_{i'})_{j'}]_j \\
&= 2\lambda^3 \left[-2c_1^3 (2t - T)^2 + 2c_1 \sum_{i,j} b^{ij} d_i d_j \right] - \lambda^3 c_0 c_1^2 (2t - T)^2 + \lambda^3 c_0 \sum_{i,j} b^{ij} d_i d_j \\
&\quad + \lambda^3 \sum_{i,j} \sum_{i',j'} b^{ij} d_i (b^{i'j'} d_{i'} d_{j'})_j - 4\lambda^3 c_1^3 (2t - T)^2 + O(\lambda^2) \\
&= (4c_1 + c_0) \lambda^3 \sum_{i,j} b^{ij} d_i d_j + \lambda^3 \sum_{i,j} \sum_{i',j'} b^{ij} d_i (b^{i'j'} d_{i'} d_{j'})_j \\
&\quad - (8c_1^3 + c_0 c_1^2) \lambda^3 (2t - T)^2 + O(\lambda^2).
\end{aligned} \tag{3.7}$$

Now we estimate $\sum_{i,j} \sum_{i',j'} b^{ij} d_i (b^{i'j'} d_{i'} d_{j'})_j$. From Condition 1.1, we get that

$$\begin{aligned}
\mu_0 \sum_{i,j} b^{ij} d_i d_j &\leq \sum_{i,j} \sum_{i',j'} \left[2b^{ij'} (b^{i'j} d_{i'})_{j'} - b_{j'}^{ij} b^{i'j'} d_{i'} \right] d_i d_j \\
&= \sum_{i,j} \sum_{i',j'} \left(2b^{ij'} b_{j'}^{i'j} d_{i'} + 2b^{ij'} b^{i'j} d_{i'j'} - b_{j'}^{ij} b^{i'j'} d_{i'} \right) d_i d_j \\
&= \sum_{i,j} \sum_{i',j'} \left(2b^{ij'} b_{j'}^{i'j} d_{i'} d_i d_j + 2b^{ij'} b^{i'j} d_{i'j'} d_i d_j - b_{j'}^{ij} b^{i'j'} d_{i'} d_i d_j \right) \\
&= \sum_{i,j} \sum_{i',j'} \left(b^{i'j'} b_{j'}^{ij} d_{i'} d_i d_j + b^{ij} b^{i'j'} d_{i'j} d_i d_j + b^{ij} b^{i'j'} d_{j'j} d_i d_{i'} \right) \\
&= \sum_{i,j} \sum_{i',j'} b^{ij} d_i (b^{i'j'} d_{i'} d_{j'})_j.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), by Condition 1.2, we obtain that

$$B \geq \lambda^3 (4c_1 + c_0) \sum_{i,j} b^{ij} d_i d_j + \lambda^3 \mu_0 \sum_{i,j} b^{ij} d_i d_j - (8c_1^3 + 2c_0 c_1^2) \lambda^3 (2t - T)^2 + O(\lambda^2)$$

$$\begin{aligned}
&\geq \lambda^3(4c_1 + c_0) \sum_{i,j} b^{ij} d_i d_j + \lambda^3 \mu_0 \sum_{i,j} b^{ij} d_i d_j - 2c_1^2(4c_1 + c_0) \lambda^3 T^2 + O(\lambda^2) \\
&\geq 2(4c_1 + c_0) \lambda^3 \left(\sum_{i,j} b^{ij} d_i d_j - c_1^2 T^2 \right) + O(\lambda^2) \\
&= 2(4c_1 + c_0) \lambda^3 (4R_1^2 - c_1^2 T^2) + O(\lambda^2).
\end{aligned}$$

Then we know that there exists a $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, we have that

$$Bv^2 \geq 8c_1(4R_1^2 - c_1^2 T^2) \lambda^3 v^2. \quad (3.9)$$

Since

$$v(0, x) = \theta(0, x)y(0, x) = 0$$

and

$$v_t(0, x) = \theta_t(0, x)y(0, x) + \theta(0, x)y_t(0, x) = 0,$$

we know that at time $t = 0$, it holds that

$$\sum_{i,j} b^{ij} l_t v_i v_j - 2 \sum_{i,j} b^{ij} l_i v_j v_t + l_t v_t^2 - \Psi v_t v + \left(Al_t + \frac{\Psi_t}{2} \right) v^2 = 0.$$

By a similar reason, we see that at time $t = T$,

$$\sum_{i,j} b^{ij} l_t v_i v_j - 2 \sum_{i,j} b^{ij} l_i v_j v_t + l_t v_t^2 - \Psi v_t v + \left(Al_t + \frac{\Psi_t}{2} \right) v^2 = 0.$$

Step 2. Integrating (2.13) in Q , taking expectation in Ω and by the argument above, we obtain that

$$\begin{aligned}
&\mathbb{E} \int_Q \theta \left\{ \left(-2l_t v_t + 2 \sum_{i,j} b^{ij} l_i v_j + \Psi v \right) \left[dy_t - \sum_{i,j} (b^{ij} y_i)_j dt \right] - \theta l_t (dy_t)^2 \right\} dx \\
&+ \lambda \mathbb{E} \int_\Sigma \sum_{i,j} \sum_{i',j'} (2b^{ij} b^{i'j'} d_{i'} v_i v_{j'} - b^{ij} b^{i'j'} d_i v_{i'} v_{j'}) \nu^j d\Sigma \\
&\geq C \mathbb{E} \int_Q \theta^2 \left[(\lambda v_t^2 + \lambda |\nabla v|^2) + \lambda^3 v^2 \right] dx dt + \mathbb{E} \int_Q \left(-2l_t v_t + 2 \sum_{i,j} b^{ij} l_i v_j + \Psi v \right)^2 dx dt.
\end{aligned} \quad (3.10)$$

Since $y = 0$ on Σ , P -a.s., from (1.9), we have

$$\begin{aligned}
&\mathbb{E} \int_\Sigma \sum_{i,j} \sum_{i',j'} \left(2b^{ij} b^{i'j'} d_{i'} v_i v_{j'} - b^{ij} b^{i'j'} d_i v_{i'} v_{j'} \right) \nu^j d\Sigma \\
&= \mathbb{E} \int_\Sigma \sum_{i,j} \sum_{i',j'} \left(2b^{ij} b^{i'j'} d_{i'} \frac{\partial v}{\partial \nu} \nu^i \frac{\partial v}{\partial \nu} \nu^{j'} - b^{ij} b^{i'j'} d_i \frac{\partial v}{\partial \nu} \nu^{i'} \frac{\partial v}{\partial \nu} \nu^{j'} \right) \nu^j d\Sigma \\
&= \mathbb{E} \int_\Sigma \left(\sum_{i,j} b^{ij} \nu^i \nu^j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu^{j'} \right) \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma
\end{aligned} \quad (3.11)$$

$$\begin{aligned}
&= \mathbb{E} \int_{\Sigma} \left(\sum_{i,j} b^{ij} \nu^i \nu^j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu^{j'} \right) \left| \theta \frac{\partial y}{\partial \nu} + y \frac{\partial \theta}{\partial \nu} \right|^2 d\Sigma \\
&= \mathbb{E} \int_{\Sigma} \left(\sum_{i,j} b^{ij} \nu^i \nu^j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu^{j'} \right) \theta^2 \left| \frac{\partial y}{\partial \nu} \right|^2 d\Sigma \\
&\leq \mathbb{E} \int_{\Sigma_0} \left(\sum_{i,j} b^{ij} \nu^i \nu^j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu^{j'} \right) \theta^2 \left| \frac{\partial y}{\partial \nu} \right|^2 d\Sigma.
\end{aligned}$$

From (3.10) and (3.11), we obtain that

$$\begin{aligned}
&\mathbb{E} \int_Q \theta \left\{ \left(-2l_t v_t + 2 \sum_{i,j} b^{ij} l_i v_j + \Psi v \right) \left[dy_t - \sum_{i,j} (b^{ij} y_i)_j dt \right] - \theta l_t (dy_t)^2 \right\} dx \\
&\quad + \lambda \mathbb{E} \int_{\Sigma_0} \left(\sum_{i,j} b^{ij} \nu^i \nu^j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu^{j'} \right) \left| \frac{\partial y}{\partial \nu} \right|^2 d\Sigma \\
&\geq C \mathbb{E} \int_Q \left[\theta^2 \left(\lambda y_t^2 + \lambda |\nabla y|^2 \right) + \lambda^3 \theta^2 y^2 \right] dx dt + \mathbb{E} \int_Q \left(-2l_t v_t + 2 \sum_{i,j} b^{ij} l_i v_j + \Psi v \right)^2 dx dt.
\end{aligned} \tag{3.12}$$

Since y solves the equation (3.4), we know that

$$\begin{aligned}
&\mathbb{E} \int_Q \theta \left\{ \left(-2l_t v_t + 2 \sum_{i,j} b^{ij} l_i v_j + \Psi v \right) \left[dy_t - \sum_{i,j} (b^{ij} y_i)_j dt \right] - \theta l_t (dy_t)^2 \right\} dx \\
&= \mathbb{E} \int_Q \theta \left\{ \left(-2l_t v_t + 2 \sum_{i,j} b^{ij} l_i v_j + \Psi v \right) \left[b_1 y_t + b_2 \cdot \nabla y + b_3 y + \chi f + \alpha \right] \right. \\
&\quad \left. - \theta l_t (b_4 y + \chi g)^2 \right\} dx dt \\
&\leq C \left\{ \mathbb{E} \int_Q \theta^2 \left[b_1 y_t + b_2 \cdot \nabla y + b_3 y + \chi f + \alpha \right]^2 + \lambda \theta^2 (b_4 y + \chi g)^2 \right\} dx dt \\
&\quad + \mathbb{E} \int_Q \left(-2l_t v_t + \sum_{i,j} b^{ij} l_i v_j + \Psi v \right)^2 dx dt \\
&\leq C \left\{ \mathbb{E} \int_Q \theta^2 (f^2 + \alpha^2 + \lambda g^2) dx dt + |b_1|_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))}^2 \mathbb{E} \int_Q \theta^2 y_t^2 dx dt + \mathbb{E} \int_Q \theta^2 b_3^2 y^2 dx dt \right. \\
&\quad \left. + |b_2|_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G,\mathbb{R}^n))}^2 \mathbb{E} \int_Q \theta^2 |\nabla y|^2 dx dt + \lambda |b_4|_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))}^2 \mathbb{E} \int_Q \theta^2 y^2 dx dt \right\} \\
&\quad + \mathbb{E} \int_Q \left(-2l_t v_t + \sum_{i,j} b^{ij} l_i v_j + \Psi v \right)^2 dx dt.
\end{aligned} \tag{3.13}$$

Recalling the definition of r_2 in (1.11), and using successively Hölder's and Sobolev's inequalities, we get

$$|b_3 \theta y|_{L^2_{\mathcal{F}}(0,T;L^2(G))}^2 \leq r_2 |\theta y|_{L^2_{\mathcal{F}}(0,T;L^s(G))}^2 \leq r_2 |\theta y|_{L^2_{\mathcal{F}}(0,T;H^{n/p})}^2 \text{ for } \frac{1}{p} + \frac{1}{s} = \frac{1}{2}. \tag{3.14}$$

For any $F \in L^2(\Omega, \mathcal{F}_T, P; H^1(\mathbb{R}^n))$, by Hölder's inequality, one has

$$\begin{aligned} |F|_{L^2(\Omega, \mathcal{F}_T, P; H^{n/p}(\mathbb{R}^n))}^2 &= \mathbb{E} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{n/p} |\hat{F}(\xi)|^{2n/p} |\hat{F}(\xi)|^{2(1-n/p)} d\xi \\ &\leq |F|_{L^2(\Omega, \mathcal{F}_T, P; H^1(\mathbb{R}^n))}^{2n/p} |F|_{L^2(\Omega, \mathcal{F}_T, P; L^2(\mathbb{R}^n))}^{2(1-n/p)}. \end{aligned}$$

Hence, we know that there is a constant $C > 0$ such that for any $\tilde{F} \in L^2(\Omega, \mathcal{F}_T, P; H_0^1(G))$, we have

$$|\tilde{F}|_{L^2(\Omega, \mathcal{F}_T, P; H^{n/p}(G))}^2 \leq C |\tilde{F}|_{L^2(\Omega, \mathcal{F}_T, P; H_0^1(G))}^{2n/p} |\tilde{F}|_{L^2(\Omega, \mathcal{F}_T, P; L^2(G))}^{2(1-n/p)}.$$

Therefore, there is a constant $C > 0$ such that for any $\overline{F} \in L^2_{\mathcal{F}}(0, T; H_0^1(G))$, it holds that

$$|\overline{F}|_{L^2_{\mathcal{F}}(0, T; H^{n/p}(G))}^2 \leq C |\overline{F}|_{L^2_{\mathcal{F}}(0, T; H_0^1(G))}^{2n/p} |\overline{F}|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^{2(1-n/p)}.$$

This, together with the inequality (3.14), implies that

$$\begin{aligned} |b_3 \theta y|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^2 &\leq C |b_3 \theta y|_{L^2_{\mathcal{F}}(0, T; H_0^1(G))}^{2n/p} |b_3 \theta y|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^{2(1-n/p)} \\ &\leq \varepsilon \lambda |b_3 \theta y|_{L^2_{\mathcal{F}}(0, T; H_0^1(G))}^2 + C(\varepsilon) r_2^{2p/(p-n)} \lambda^{-n/(p-n)} |b_3 \theta y|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^2, \end{aligned} \quad (3.15)$$

where ε is small enough and $C(\varepsilon)$ depends on ε .

Taking $\lambda_2 = C(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1) \geq \max\{\lambda_0, \lambda_1\}$, combining (3.12), (3.13) and (3.15), for any $\lambda \geq \lambda_2$, we have that

$$\begin{aligned} &C \lambda \mathbb{E} \int_{\Sigma_0} \theta^2 \left(\sum_{i,j} b^{ij} \nu_i \nu_j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu_{j'} \right) \left| \frac{\partial y}{\partial \nu} \right|^2 d\Sigma + C \mathbb{E} \int_Q \theta^2 (f^2 + \alpha^2 + \lambda g^2) dx dt \\ &\geq \mathbb{E} \int_Q \theta^2 \left(\lambda y_t^2 + \lambda |\nabla y|^2 + \lambda^3 y^2 \right) dx dt. \end{aligned} \quad (3.16)$$

Recalling the property of χ (see (3.3)) and $y = \chi z$, from (3.16), we find

$$\begin{aligned} &C \lambda \mathbb{E} \int_{\Sigma_0} \left(\sum_{i,j} b^{ij} \nu_i \nu_j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu_{j'} \right) \theta^2 \left| \frac{\partial z}{\partial \nu} \right|^2 d\Sigma + C \mathbb{E} \int_Q \theta^2 (f^2 + \lambda g^2) dx dt \\ &+ C(r_1 + 1) \left[\mathbb{E} \int_0^{\frac{T}{2} - \varepsilon_1 T} \int_G \theta^2 (z_t^2 + |\nabla z|^2 + z^2) dx dt + \mathbb{E} \int_{\frac{T}{2} + \varepsilon_1 T}^T \int_G \theta^2 (z_t^2 + |\nabla z|^2 + z^2) dx dt \right] \\ &\geq \mathbb{E} \int_{\frac{T}{2} - \varepsilon_0 T}^{\frac{T}{2} + \varepsilon_0 T} \int_G \theta^2 \left(\lambda z_t^2 + \lambda |\nabla z|^2 + \lambda^3 z^2 \right) dx dt. \end{aligned} \quad (3.17)$$

Combining (2.3) and (3.17), we know that there is a $\lambda_3 = C(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1) \geq \lambda_2$ such that for all $\lambda \geq \lambda_3$, it holds that

$$\begin{aligned} &C \lambda \mathbb{E} \int_{\Sigma_0} \theta^2 \left(\sum_{i,j} b^{ij} \nu_i \nu_j \right) \left(\sum_{i',j'} b^{i'j'} d_{i'} \nu_{j'} \right) \left| \frac{\partial z}{\partial \nu} \right|^2 d\Sigma + C \mathbb{E} \int_Q \theta^2 (f^2 + \lambda g^2) dx dt \\ &\geq \mathbb{E} \int_G \theta^2(0) \left(\lambda z_1^2 + \lambda |\nabla z_0|^2 + \lambda^3 z^2(0) \right) dx dt. \end{aligned} \quad (3.18)$$

Taking $\lambda = \lambda_3$, we obtain that

$$\begin{aligned} & C e^{\lambda_3 R_1^2} \left[\mathbb{E} \int_{\Sigma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Sigma + \mathbb{E} \int_Q (f^2 + g^2) dx dt \right] \\ & \geq e^{-\frac{T^2}{4} \lambda_3} \mathbb{E} \int_G (z_1^2 + |\nabla z_0|^2) dx dt. \end{aligned} \quad (3.19)$$

This leads to the inequality (1.12) immediately.

4 Proof of Theorem 1.2

This section is devoted to a proof of Theorem 1.2.

Proof of Theorem 1.2: Let $h_0 \in C^1(\overline{G}; \mathbb{R}^n)$ such that $h_0 = \nu$ on Γ , and let $\rho \in C^2(\overline{G}; [0, 1])$ such that

$$\begin{cases} \rho = 1 & \text{in } \mathcal{O}_{\frac{\delta}{3}}(\Gamma_0), \\ \rho = 0 & \text{in } G \setminus \mathcal{O}_{\frac{\delta}{2}}(\Gamma_0). \end{cases} \quad (4.1)$$

Let $h = \rho \theta^2 h_0$ in the equality (2.2), noting that $y_j = \frac{\partial y}{\partial \nu} \nu^j$ on Σ , by integrating by parts, we see

$$\begin{aligned} & \mathbb{E} \int_{\Sigma} \left(\sum_{i,j=1}^n b^{ij} \nu^i \nu^j \right) \rho \theta^2 \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \\ &= \mathbb{E} \int_Q \sum_{i=1}^n \left[2(h \cdot \nabla y) \sum_{j=1}^n b^{ij} y_j + h^i \left(y_t^2 - \sum_{j,k=1}^n b^{jk} y_j y_k \right) \right]_i dx dt \\ &= -\mathbb{E} \int_Q 2 \left[\left(dy_t - \sum_{i,j=1}^n (b^{ij} y_i)_j dt \right) h \cdot \nabla y - d(y_t h \cdot \nabla y) + y_t h_t \nabla y dt \right. \\ & \quad \left. - \sum_{i,j,k=1}^n b^{ij} y_i y_k h_j^k dt - \operatorname{div}(h) y_t^2 dt + \sum_{i,j=1}^n y_j y_i \operatorname{div}(b^{ij} h) \right] dx dt \\ &\leq C \left\{ \frac{1}{\lambda} \mathbb{E} \int_Q \theta^2 (b_1 y_t + b_2 \cdot \nabla y + b_3 y + \chi f + \alpha)^2 dx dt \right. \\ & \quad \left. + \lambda \mathbb{E} \int_0^T \int_{\mathcal{O}_{\frac{\delta}{2}}(\Gamma_0)} \theta^2 (y_t^2 + |\nabla y|^2) dx dt \right\}. \end{aligned} \quad (4.2)$$

Now let us deal with the term $\mathbb{E} \int_0^T \int_{\mathcal{O}_{\frac{\delta}{2}}(\Gamma_0)} \theta^2 |y_t|^2 dx dt$. Let $\rho_1 \in C^2(\overline{G}; [0, 1])$ satisfying that

$$\begin{cases} \rho_1 = 1 & \text{in } \mathcal{O}_{\frac{\delta}{2}}(\Gamma_0), \\ \rho_1 = 0 & \text{in } G \setminus \mathcal{O}_{\delta}(\Gamma_0). \end{cases}$$

Put $\eta = \rho_1^2 \theta^2$. By virtue of that y solves the equation (3.4), we have

$$\begin{aligned}
& \mathbb{E} \int_Q \eta y (b_1 y_t + b_2 \cdot \nabla y + b_3 y + f + \alpha) dx dt \\
&= \mathbb{E} \int_Q \eta y \left[dy_t - \sum_{i,j=1}^n (b^{ij} y_i)_j dt \right] dx \\
&= -\mathbb{E} \int_Q \left[y_t (\eta_t y + \eta y_t) \right] dx dt + \mathbb{E} \int_Q \eta \sum_{i,j=1}^n b^{ij} y_i y_j dx dt \\
&\quad + \mathbb{E} \int_Q y \sum_{i,j=1}^n b^{ij} y_i \eta_j dx dt,
\end{aligned} \tag{4.3}$$

this implies that

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathcal{O}_{\frac{\delta}{2}}(\Gamma_0)} \theta^2 |y_t|^2 dx dt \\
&\leq C \left\{ \frac{1}{\lambda^2} \mathbb{E} \int_Q \theta^2 [b_1 y_t + b_2 \cdot \nabla y + b_3 y + \chi f + \alpha]^2 dx dt \right. \\
&\quad \left. + \mathbb{E} \int_0^T \int_{\mathcal{O}_{\delta}(\Gamma_0)} \theta^2 (\lambda^2 y^2 + |\nabla y|^2) dx dt \right\}.
\end{aligned} \tag{4.4}$$

From (3.16), (4.3) and (4.4), we get that there is a $\lambda_4 = C(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1) > 0$ such that for any $\lambda \geq \max\{\lambda_3, \lambda_4\}$, it holds that

$$\begin{aligned}
& \mathbb{E} \int_G \theta^2(0) \left[\lambda (|\nabla z_0|^2 + |z_1|^2) + \lambda^3 |z_0|^2 \right] dx \\
&\leq C \mathbb{E} \int_0^T \int_{\mathcal{O}_{\delta}(\Gamma_0)} \theta^2 (\lambda^3 z^2 + \lambda |\nabla z|^2) dx dt.
\end{aligned} \tag{4.5}$$

Since

$$e^{-\lambda \frac{T^2}{4}} \leq \theta(t, x) \leq e^{\lambda \frac{T^2}{4}},$$

we see that

$$\mathbb{E} \int_G (|\nabla z_0|^2 + |z_1|^2 + |z_0|^2) dx \leq C e^{\frac{\lambda T^2}{2}} \mathbb{E} \int_0^T \int_{\mathcal{O}_{\delta}(\Gamma_0)} (|\nabla z|^2 + z^2) dx dt. \tag{4.6}$$

This, together with Poincaré's inequality, implies the inequality (1.13) immediately.

5 A state observation problem

This section is addressed to a state observation problem for semilinear stochastic hyperbolic equations. Let

$$F(\eta, \varrho, \zeta) : \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{R}^1$$

and

$$K(\eta) : \mathbb{R}^1 \rightarrow \mathbb{R}^1$$

be two known nonlinear functions. Consider the following semilinear stochastic hyperbolic equation

$$\begin{cases} dw_t - \sum_{i,j} (b^{ij} w_i)_j dt = F(w, w_t, \nabla w) dt + K(w) dB(t) & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } G, \end{cases} \quad (5.1)$$

where the initial data $(w_0, w_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$ are unknown random variables.

We put the following assumption:

(AS) The nonlinear functions $F(\cdot, \cdot, \cdot)$ and $K(\cdot)$ satisfy the following:

1.

$$|F(\eta_1, \varrho, \zeta) - F(\eta_2, \varrho, \zeta)| + |K(\eta_1) - K(\eta_2)| \leq L(1 + |\eta_1|^{p-1} + |\eta_2|^{p-1})|\eta_1 - \eta_2| \\ \forall \eta_1, \eta_2, \varrho \in \mathbb{R}^1, \zeta \in \mathbb{R}^n$$

with $1 \leq p \leq \frac{n}{n-2}$ if $n \geq 3$; $1 \leq p < \infty$ if $n = 1, 2$, for some constant $L > 0$;

2.

$$|F(\eta, \varrho_1, \zeta_1) - F(\eta, \varrho_2, \zeta_2)| \leq L(|\varrho_1 - \varrho_2| + |\zeta_1 - \zeta_2|) \\ \forall (\eta, \varrho_i, \zeta_i) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^n, i = 1, 2, \\ |F(0, \varrho, \zeta)| \leq L(|\varrho| + |\zeta|) \quad \forall (\varrho, \zeta) \in \mathbb{R}^1 \times \mathbb{R}^n$$

for some constant $L > 0$;

3. for any given initial data $(w_0, w_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$, (5.1) admits a unique solution $w = w(\cdot; w_0, w_1) \in H_T$ (the solution of (5.1) is defined similarly to the one of (1.2)).

Here since we do not introduce any sign condition on the nonlinear functions $F(\cdot, \cdot, \cdot)$ and $K(\cdot)$, the global existence of a solution to (5.1) is not guaranteed. This is why we need to impose the third assumption in **(AS)**.

The state observation problem associated to the equation (5.1) is as follows.

• **Identifiability.** Is the solution $w \in H_T$ (to (5.1)) determined uniquely by the observation $\frac{\partial w}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ (resp. $w|_{(0,T) \times \mathcal{O}_\delta(\Gamma_0)}$)?

• **Stability.** Assume that two solutions w and \hat{w} (to the equation (5.1)) are given. Let $\frac{\partial w}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ (resp. $w|_{(0,T) \times \mathcal{O}_\delta(\Gamma_0)}$) and $\frac{\partial \hat{w}}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ (resp. $\hat{w}|_{(0,T) \times \mathcal{O}_\delta(\Gamma_0)}$) be the corresponding observations. Can we find a positive constant C such that

$$|w - \hat{w}| \leq C \left\| \frac{\partial w}{\partial \nu} - \frac{\partial \hat{w}}{\partial \nu} \right\| \left(\text{resp. } |w - \hat{w}| \leq C \|w - \hat{w}\| \right),$$

with appropriate norms in both sides?

- **Reconstruction.** Is it possible to reconstruct $w \in H_T$ to (5.1), in some sense, from the observation $\frac{\partial w}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ (*resp.* $w|_{(0,T) \times \mathcal{O}_\delta(\Gamma_0)}$)?

The state observation problem for systems governed by deterministic partial differential equations is studied extensively (see [7, 11, 20] and the rich references therein). However, the stochastic case attracts very little attention. To our best knowledge, [25, 16] are the only two published papers addressing this topic. In [16], the author studied the state observation problem for stochastic Schrödinger equations via the Carleman estimate for the equation. In [25], the author addressed the state observation problem for stochastic wave equations and proved the following result

$$\begin{aligned} & |(w(t) - \hat{w}(t), w_t(t) - \hat{w}_t(t))|_{L^2(\Omega, \mathcal{F}_t, P; H_0^1(G) \times L^2(G))} \\ & \leq e^{Ct^{-1}} \tilde{C} \left| \frac{\partial w}{\partial \nu} - \frac{\partial \hat{w}}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))} \quad \text{for any } t > 0. \end{aligned} \quad (5.2)$$

Obviously, one cannot let $t = 0$ in (5.2), which means that the initial state cannot be obtained from the observation. In this paper, by means of Theorem 1.1, we can give positive answers to the above first and second questions, that is, we prove that the whole state can be observed by the boundary or internal observation.

First, thanks to the Sobolev embedding theorem and the conditions on $F(\cdot, \cdot, \cdot)$ and $K(\cdot)$, we know

$$F(w, w_t, \nabla w) \in L^2_{\mathcal{F}}(0, T; L^2(G)), \quad K(w) \in L^2_{\mathcal{F}}(0, T; L^2(G))$$

for any $w \in H_T$. Thus, by Proposition 2.1, we know $\frac{\partial w}{\partial \nu} \in L^2_{\mathcal{F}}(0, T; L^2(G_0))$. Now, we define two nonlinear maps \mathcal{M}_1 and \mathcal{M}_2 as follows:

$$\begin{cases} \mathcal{M}_1 : L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G)) \rightarrow L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)), \\ \mathcal{M}_1(w_0, w_1) = \frac{\partial w}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}, \\ \mathcal{M}_2 : L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G)) \rightarrow L^2_{\mathcal{F}}(0, T; L^2(\mathcal{O}_\delta(\Gamma_0))), \\ \mathcal{M}_2(w_0, w_1) = \nabla w|_{(0,T) \times \mathcal{O}_\delta(\Gamma_0)}, \end{cases}$$

where w solves the equation (5.1).

We have the following result.

Theorem 5.1 *Let Condition (1.1) and Condition (1.2) be satisfied. There exists a constant $\tilde{C} = \tilde{C}(L, T, G, (b^{ij})_{1 \leq i, j \leq n}, \Gamma_0, \delta) > 0$ such that for any initial data $(w_0, w_1), (\hat{w}_0, \hat{w}_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$, it holds that*

$$|(w_0 - \hat{w}_0, w_1 - \hat{w}_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} \leq \tilde{C} |\mathcal{M}_1(w_0, w_1) - \mathcal{M}_1(\hat{w}_0, \hat{w}_1)|_{L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))} \quad (5.3)$$

and

$$|(w_0 - \hat{w}_0, w_1 - \hat{w}_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} \leq \tilde{C} |\mathcal{M}_2(w_0, w_1) - \mathcal{M}_2(\hat{w}_0, \hat{w}_1)|_{L^2_{\mathcal{F}}(0, T; L^2(\mathcal{O}_\delta(\Gamma_0)))}, \quad (5.4)$$

where $\hat{w} = \hat{w}(\cdot; \hat{w}_0, \hat{w}_1) \in H_T$ is the solution to (5.1) with (w_0, w_1) replaced by (\hat{w}_0, \hat{w}_1) .

Remark 5.1 *Theorem 5.1 indicates that the state $w(t)$ of (5.1) (for $t \in [0, T]$) can be uniquely determined from the observed boundary data $\frac{\partial w}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ or $\nabla w|_{(0,T) \times \mathcal{O}_\delta(\Gamma_0)}$, P -a.s., and continuously depends on it. Therefore, we answer the first and second questions for the state observation problem of the system (5.1) positively.*

Proof of Theorem 5.1: Set

$$y = \hat{w} - w.$$

It is easy to see that y is a solution of (5.1) with

$$\begin{cases} b_1 = \int_0^1 \partial_\eta F(w + s(\hat{w} - w), w_t, \nabla w) ds, & b_2 = \int_0^1 \partial_\varrho(\hat{w}, w_t + s(\hat{w}_t - w_t), \nabla w) ds, \\ b_3 = \int_0^1 \partial_\zeta F(\hat{w}, \hat{w}_t, \nabla w + s(\nabla \hat{w} - \nabla w)) ds, & b_4 = \int_0^1 \partial_\eta K(w + s(\hat{w} - w)) ds. \end{cases}$$

Then, the inequality (5.3) follows from Theorem 1.1 and the inequality (5.4) comes from Theorem 1.2. \square

As a direct consequence of Theorem 5.1, we have the following unique continuation property for the equation (1.2).

Corollary 5.1 *Let Condition (1.1) and Condition (1.2) be satisfied. Assume that $f = g = 0$ in Q , P -a.s. If a solution of the equation (1.2) satisfies $y = 0$ in $(0, T) \times \mathcal{O}_\delta(\Gamma_0)$, P -a.s., then we have that $y = 0$ in Q , P -a.s.*

Remark 5.2 *The analogous result of Corollary 5.1 for deterministic hyperbolic equations with nonsmooth lower order terms was first obtained in [21].*

Due to the need from Control/Inverse Problems of partial differential equations, the study of the global unique continuation for partial differential equations is very active (see [3, 21, 26] and the references therein) in recent years. Compared with the plentiful studying of the unique continuation property for partial differential equations, the study for stochastic partial differential equations is cold and cheerless. To the best of our knowledge, [23, 24, 16, 17] are the only published articles which concern this topic, and the above unique continuation property for stochastic hyperbolic equations has not been presented in the literature.

Next, we consider the reconstruction of the state w . Denote by φ the observation on $(0, T) \times \Gamma_0$ and by

$$W \triangleq \{w \in H_T : w \text{ solves (5.1) for some initial data } (w_0, w_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))\}.$$

Put

$$J_1(w) = \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial w}{\partial \nu} - \varphi \right|^2 d\Gamma dt \quad \text{for } w \in W.$$

Let \tilde{w} be the state of (5.1) corresponding to the observation φ . Then, it is clear that the state \tilde{w} satisfies that

$$J_1(\tilde{w}) = \min_{w \in W} J_1(w) = 0.$$

Hence, the construction of \tilde{w} can be converted to the study of the following optimization problem

(P₁) Find a $w \in W$ which minimizes $J_1(\cdot)$.

By a similar argument, we can show the construction of \tilde{w} can be deduced to the following optimization problem

(P₂) Find a $w \in W$ which minimize $J_2(w) = \int_0^T \int_{G_0} |\nabla w - \psi|^2 dx dt$,

where ψ is the internal observation.

To give efficient algorithms to solve problem (P₁) and (P₂) is beyond the scope of this paper and will be studied in our forthcoming paper.

6 Further comments and open problems

There are plenty of open problems in the topic of this paper. Some of them are particularly relevant and could need new ideas and further developments.

- **Efficiency algorithm for the construction of the solution w from the observation**

In this paper, we only answer the first and the second questions in the state observation problem. The third one is still open. Due to the stochastic feature, some efficient approaches for hyperbolic equation(see [11] for example), become invalid. In the end of Section 5, we show that it can be solved by studying an optimization problem. In this context, it seems that one may utilize the great many sharp methods from optimization theory to study the construction of (w_0, w_1) . However, thanks to the stochastic setting, it seems that one cannot simply mimic these methods. A detailed study of this interesting but difficult problem is beyond the scope of this paper.

- **Observability estimate and unique continuation property with less restrictive conditions**

In this paper, we prove the inequality (1.12) and (1.13) under the Condition (1.1) and Condition (1.2). It is well known that a sharp sufficient condition for establishing observability estimate for deterministic hyperbolic equations with time invariant lower order terms is that the triple $(G, \Gamma_0, T)((G, \mathcal{O}_\delta(\Gamma_0), T))$ satisfies the geometric optic condition introduced in [2]. It would be quite interesting and challenging to extend this result to the stochastic setting. However, there are lots of things should be done before solving this problem. For instance, the propagation of singularities for stochastic partial differential equations, at least, for stochastic hyperbolic equations, should be established.

As we have pointed out in Remark 1.6, it is more interesting to get the following inequality

$$\begin{aligned} & |(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; L^2(G) \times H^{-1}(G))} \\ & \leq e^{C(r_1^2 + r_2^{\frac{1}{3/2-n/p}+1})} \left(|z|_{L^2_{\mathcal{F}}(0, T; L^2(\mathcal{O}_\delta(\Gamma_0)))} + |f|_{L^2_{\mathcal{F}}(0, T; L^2(G))} + |g|_{L^2_{\mathcal{F}}(0, T; L^2(G))} \right). \end{aligned} \quad (6.1)$$

For deterministic hyperbolic equations, the inequality (6.1) can be obtained by combining the global Carleman estimate and the multiplier method (see [5] for example). If one follows the method to study the stochastic problem, one will meet some real difficulty. In fact, as the deterministic settings, from Theorem 1.1, by means of a suitable choice of multiplier, one can get

$$\begin{aligned} & |(z_0, z_1)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))} \\ & \leq C e^{C(r_1^2 + r_2^{\frac{1}{3/2 - n/p} + 1})} \left(\mathbb{E} \int_0^T \int_{\mathcal{O}_\delta(\Gamma_0)} (z_t^2 + z^2) dx dt + |f|_{L^2_{\mathcal{F}}(0, T; L^2(G))} + |g|_{L^2_{\mathcal{F}}(0, T; L^2(G))} \right). \end{aligned} \quad (6.2)$$

Then, by employing the Compact/Uniqueness argument, we can eliminate the term “ z^2 ” in the right hand side of (6.2) for deterministic case. A key point in the Compact/Uniqueness argument is the fact that $H^1((0, T) \times G)$ is compactly imbedded into $L^2((0, T) \times G)$. However, the corresponding result is not true in the stochastic settings. One can easily show that even H_T (recall that H_T is given as in (1.5)) is not compact embedded in $L^2_{\mathcal{F}}(0, T; L^2(G))$. The missing of compactness leads to new difficulty for establishing internal observability estimate for stochastic hyperbolic equations.

Under the Condition (1.1) and Condition (1.2), $y = 0$ in Q , P -a.s., provided that $y = 0$ in $(0, T) \times \mathcal{O}_\delta(\Gamma_0)$. Compared to the classical unique continuation result for deterministic hyperbolic equations (see [19] for example), the conditions in this paper is very restrictive. It would be quite interesting but maybe challenging to prove whether these results in [19] is true or not for stochastic hyperbolic equations.

• Some other inverse problems for stochastic hyperbolic equations

In this paper, we show that the state can be uniquely determined by the observation via Carleman estimate. For deterministic partial differential equations, there are lots of other interesting inverse problems solved by some methods based on Carleman estimate. For example, the multidimensional coefficient/source inverse problems with single measurement data. Both the global uniqueness and global stability are obtained by some methods which are mainly based on Carleman estimates. There are so many works in this topic. Hence, we do not list them comprehensively and we refer the readers to two nice surveys [20, 8] and the rich references therein. One will meet substantially new difficulties in the study of inverse problems for stochastic partial differential equations. For instance, unlike the deterministic partial differential equations, the solution of a stochastic partial differential equations is usually non-differentiable with respect to the variable with noise (say, the time variable considered in this paper). Also, the usual compactness embedding result does not remain true for the solution spaces related to stochastic partial differential equations. Due to these new phenomena, most of the powerful methods in [20, 8] cannot be applied to stochastic partial differential equations directly. In [15], the author studied an inverse source problem for stochastic parabolic equations involved in some special domain. It seems that it is hard to generalize the method in [15] for the study of stochastic partial differential equations in general domains.

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